VORTEX NEAR A FREE SURFACE

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In the basic treatments of the theory of the immersed wing, it is assumed that the free surface deviates little from the plane horizontal surface of the undisturbed fluid [1,2]. Such an approach renders this theory inherently inapplicable to the case of an immersed wing close to the free surface. Therefore, it is useful to investigate the characteristics of the case of small depths of immersion by means of an exact solution even of a simplified problem.

We shall present the solution of the problem of the two-dimensional flow past a vortex of an ideal, incompressible, weightless fluid which has a free surface. Underneath, the flow is bounded by a solid horizontal wall.

Hopkinson [3] was the first to investigate the theory of potential flows containing singularities. The problems which are closest to the one under investigation here were solved by Simmons [4] and Nikol'skii [5].

1. General solution of the problem. Let w be a complex potential and $\zeta = dw/v_0dz$ the nondimensional complex potential velocity of the vortex located at a depth h below the free surface (Fig. 1). Let the rate of flow of the fluid be denoted by $q/2 = lv_0$, where v_0 is the velocity and l is the depth of a stream of infinity. It may be assumed that a general solution is obtained if the conformal transformation of the field of variation of w and ζ into any canonical region of variation of the parametric variable τ is established. Let us choose this region to be the upper half-circle of unit radius with the center at the origin of coordinates (Fig. 2).

If the singularities and zeros of the functions $w(\tau)$ and $\zeta(\tau)$ are known in a unit half-circle, then by making use of the whole of the plane which also contains the image of the transformation of $w(\tau)$ and $\zeta(\tau)$, all of its singularities and zeros may be determined. Thereupon the functions $w(\tau)$ and $\zeta(\tau)$ may easily be constructed [6].

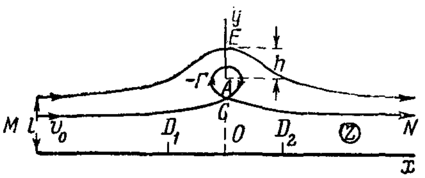

Fig. 1.

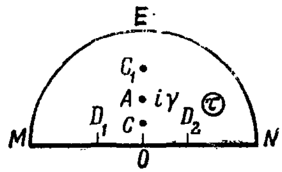

Fig. 2.

Let us begin with the function $w(\tau)$. This function has logarithmic singularities at the point $A(\tau = i\gamma)$, where a vortex with circulation Γ is located, and at the points $M(\tau = -1)$ and $N(\tau = 1)$, at which sources are located. Upon extending the function $w(\tau)$ over the whole of the plane of the variable τ we readily find that there are logarithmic singularities also at the points $\tau = -i\gamma^{-1}$ (circulation $-\Gamma$) and $\tau = -i\gamma$, $\tau = i\gamma^{-1}$ (circulation Γ). The function $w(\tau)$ has the form

$$w = \frac{q}{\pi} \ln \frac{1+\tau}{1-\tau} + \frac{i\Gamma}{2\pi} \ln \frac{(\tau - i\gamma)(\tau + i\gamma^{-1})}{(\tau + i\gamma)(\tau - i\gamma^{-1})} \tag{1.1}$$

By direct verification it is seen readily that: (1) at the point $\tau = i\gamma$ we have a vortex with circulation $-\Gamma$; (2) along the unit half-circle (free surface) $\psi = 1/2\, q$ and along the real diameter (bottom) $\psi = 0$. From (1.1) we obtain

$$\frac{dw}{d\tau} = \frac{[2q - \Gamma(\gamma^{1} - \gamma)](\tau^4 + 2\varkappa\tau^2 + 1)}{\pi(1-\tau^2)(\tau^2+\gamma^2)(\tau^2+\gamma^{-2})} \tag{1.2}$$

where

$$2\varkappa = \frac{\gamma^{-2} + \gamma^2 + \Gamma q^{-1}(\gamma^{-1} - \gamma)}{1 - {}^1/_2 \Gamma q^{-1}(\gamma^{-1} - \gamma)}$$

Since the reflection of the region of flow z onto the interior of a unit half-circle is conformal, then inside this half-circle dz/dt is neither zero nor infinite. Hence we have at the critical points $dw/d\tau = 0$. Let us find the values of τ at the critical points. They are obtained from the solution of the equation

$$\tau^4 + 2\varkappa\tau^2 + 1 = 0$$

we have

$$\tau_{1,2}^2 = -\varkappa + \sqrt{\varkappa^2 - 1} = -\frac{1}{\sqrt{\varkappa^2 - 1} + \varkappa} \tag{1.3}$$

$$\tau_{3,4}^{2} = -\varkappa - \sqrt{\varkappa^2 - 1} = \frac{1}{\sqrt{\varkappa^2 - 1} - \varkappa}$$

First of all we note that $\kappa^2 > 1$, because otherwise $|\tau_k| = 1$, and we would have critical points on the free surface, which is impossible.

Consider the case $\kappa > 0$. Here $|\tau_{1,2}| < 1$ and $|\tau_{3,4}| > 1$. Inside the stream there exists only one critical point C, to which there corresponds in the region τ the point (Figs. 1 and 2)

$$\tau_1 = \delta = i\sqrt{\varkappa - \sqrt{\varkappa^2 - 1}} = \frac{i}{\sqrt{\varkappa + \sqrt{\varkappa^2 - 1}}} \tag{1.4}$$

The point conjugate to the point C in region τ has the affix $\tau_2 = \bar{\delta} = -\delta$. It is also evident that $\tau_3 = \delta^{-1} = -\tau_4$.

Contrary to this, in the case $\kappa < 0$, $|\tau_{3,4}| < 1$ and $|\tau_{1,2}| > 1$. In this case there exist on the boundary of the flow, that is, on the solid wall, two critical points

$$\tau_3 = \frac{1}{\sqrt{\sqrt{\varkappa^2 - 1} - \varkappa}} = \delta, \qquad \tau_4 = -\delta \tag{1.5}$$

In addition, we note that $\tau_1 = -\tau_2 = \delta^{-1}$.

In the transitional case $\kappa = \infty$ there exists one critical point 0 corresponding to $\tau = 0$. Using the above observations we can easily construct the function ζ. Besides its zeros at the critical points the function $\zeta = dw/v_0 dz$ has a pole at the point $A(\tau = i\gamma)$. After continuation across the real axis τ, along which $\operatorname{Im}\zeta = 0$, the zeros correspond to the zeros of the function ζ at the conjugate points and mutually also the poles. Along the curved line $|\tau| = 1$, along which $|\zeta| = 1$, the zeros of the function ζ become the poles and the poles become zeros. We obtain as a result

$$\zeta = \frac{dw}{v_0 dz} = \frac{(\tau^2 - \delta^2)(\gamma^2\tau^2 + 1)}{(1 - \tau^2\delta^2)(\tau^2 + \gamma^2)} \tag{1.6}$$

It is not difficult to see by a direct check that $\zeta(\tau)$ has the corresponding singularities along the bottom $\operatorname{Im}\zeta = 0$, on the free surface $|\zeta| = 1$ and $\zeta = 1$ for $\tau = \pm 1$. The quantity δ is determined for $\kappa > 0$ from (1.4) and for $\kappa < 0$ from (1.5).

Equations (1.6) and (1.1) or (1.2) give a general solution of the problem.

2. Immersion depth of a vortex. The function $z(\tau)$ is easily found from (1.6) and (1.2)

$$z(\tau) = \frac{1}{v_0}\int \frac{v_0 dz}{dw}\frac{dw}{d\tau}\, d\tau = \frac{[2q - \Gamma(\gamma^{-1} - \gamma)]\,\delta^2}{\pi v_0 \gamma^2}\int \frac{(\tau^2 - \gamma^{-2})^2\, d\tau}{(\tau^2 + \gamma^{-2})^2 (1 - \tau^2)} \qquad (2.1)$$

The integral of (2.1) may be calculated in terms of elementary functions

$$z = -\frac{[2q - \Gamma(\gamma^{-1} - \gamma)]\,\delta^2}{v_0 \pi \gamma^2}\left[\frac{B\gamma^2}{2}\frac{\tau}{\tau^2 + \gamma^{-2}} + \frac{C\gamma}{2i}\ln\frac{i\gamma^{-1} - \tau}{i\gamma^{-1} + \tau} + \frac{A}{2}\ln\frac{1 + \tau}{1 - \tau}\right] \qquad (2.2)$$

where

$$A = \frac{(1 - \delta^{-2})^2}{(1 + \gamma^{-2})^2}, \qquad B = \frac{(\gamma^{-2} + \delta^{-2})^2}{1 + \gamma^{-2}}, C = \frac{(\gamma^{-2} + \delta^{-2})}{2(1 + \gamma^{-2})^2}, \qquad \left[3\left(\frac{1}{\delta^2} - 1\right) - \frac{1}{\gamma^2} + \frac{\gamma^2}{\delta^2}\right]$$

The constant of integration is chosen in such a way that $z = 0$ for $\tau = 0$ (Figs. 1 and 2).

The immersion depth of a vortex under the free surface equals

$$h = \frac{z(i) - z(i\gamma)}{i} = -\frac{[2q - \Gamma(\gamma^{-1} - \gamma)]\,\delta^2}{v_0 \pi \gamma^2}\left[\frac{B\gamma^4}{2}\frac{1 + \gamma^2 - \gamma}{1 - \gamma^4} - \right.$$

$$\left. - \frac{C\gamma}{2}\ln\frac{1 + \gamma^2}{(1 + \gamma)^2} + A\left(\frac{\pi}{4} - \tan^{-1}\gamma\right)\right] \qquad (2.3)$$

At a first glance at Figs. 1 and 2 it may appear that for $\gamma \to 1$ the immersion depth of the vortex $h \to 0$. But this is not so. There exists a minimum possible immersion depth of the vortex h_{min}. A vortex cannot rise higher without disrupting the pattern of potential flow. In the case of a real submerged wing this initiates the aspiration of air inside, i.e. the appearance of cavitation as a wing comes close to the free surface. Assume $\gamma = 1 - \varepsilon$, where ε is an infinitely small positive quantity. Then we obtain asymptotic equations

$$\delta \approx i\left(1 - \sqrt{\frac{\Gamma\varepsilon}{q}}\right), \qquad -\frac{[2q - \Gamma(\gamma^{-1} - \gamma)]\,\delta^2}{v_0 \pi \gamma^2} \approx \frac{2q}{v_0 \pi}$$

$$B \approx 2\Gamma\varepsilon / q, \qquad A \approx 1, \qquad C \approx 2\sqrt{\Gamma\varepsilon / q}$$

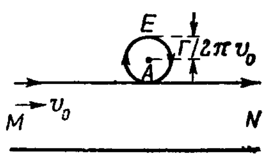

Fig. 3.

After substituting these quantities into (2.3) and going to the limit, we find that

$$h_{\min} = \frac{\Gamma}{2\pi v_0} \qquad (2.4)$$

Note that the quantity h_{min} does not depend on the rate of flow $q/2$.

3. Lifting force of the vortex. By applying the theorem of the projection of the change of rate of momentum it is easily verified that the vortex in a weightless fluid has only a lifting force Y and does not have a drag force, hence $X = 0$. The lifting force of the vortex may be calculated by various methods without much difficulty. In particular, since the stream lines which approach a vortex indefinitely closely become infinitely small closed contours which encompass the vortex, the force acting on the vortex may be calculated through an integral along an infinitely small contour

$$X + iY = \frac{i\rho}{2} \oint \left(v_0^2 - \frac{dw}{dz}\frac{\overline{dw}}{dz}\right) dz = -\frac{i\rho}{2} \oint \overline{\frac{dw}{dz} dw} \qquad (3.1)$$

As a result of these computations we find

$$Y = \frac{\rho v_0 [2q - \Gamma(\gamma^{-1} - \gamma)](\gamma^2 + \delta^2)}{4\delta^2\gamma(1 + \gamma^2)^2} [\delta^2(1 + 3\gamma^2) - \gamma^2(3 + \gamma^2)] \qquad (3.2)$$

It is easily verified that at minimum immersion ($\gamma \to 1$) the lifting force vanishes, since $\lim(\gamma^2 + \delta^2) = 0$, for $\gamma \to 1$ and the remaining factors are finite.

According to an interesting comment by S.S. Grigorian in the limiting case $\gamma = 1$ we have the configuration of the flow represented in Fig. 3.

4. The limiting case of infinitely large depth. The limit of the general expressions of Section 1 to the case of infinitely large depth entails known inconveniences, because in the case of an infinitely large depth in a stream we have only one point at infinity, while in the region of the variable τ the points at infinity M and N are separated by a finite interval. Passage to the limiting case of an infinite depth, however, is easily accomplished by replacing the parametric variable τ by a new parametric variable u varying in the lower half-plane. A particular solution for the case of infinite depth has the form

$$\frac{dw}{v_0 dz} = \frac{i - u}{i + u} \frac{i\sqrt{1 + \Gamma/Q} + u}{i\sqrt{1 + \Gamma/Q} - u}, \qquad \frac{dw}{du} = \frac{Q}{\pi}\frac{u^2 + 1 + \Gamma/Q}{u^2 + 1} \qquad (4.1)$$

The reflection onto the lower half-plane is chosen in such a way that the point $u = -i$ corresponds to the vortex (Fig. 4) and the points M and N go to infinity. The solution in the case of infinite depth using reflection on the inside of the circle may be obtained readily from the solution of Nikol'skii's problem [5] of a vortex in a jet of finite thickness.

The lifting force of a vortex in an infinitely deep weightless fluid is determined in the following manner:

$$Y = \rho v_0 \Gamma \frac{2}{\sqrt{1 + \Gamma/Q} + 1 + \Gamma/Q + 1}$$

For an infinite immersion depth of the vortex $\Gamma/Q = 0$ and $Y = \rho v_0 \Gamma$.

In the case of minimum depth ($\Gamma/Q \to \infty$) the lifting force vanishes.

5. The case of downward force. Let us consider another case when the circulation of a vortex is directed counterclock-wise, i.e. opposite to what is shown in Fig. 1. The pattern of resulting flow is represented in Fig. 5. It is noted that the critical C_1 will be higher than point A even in the field of the parametric variable τ (see Fig. 2). As before, the complex velocity will be expressed by equation (1.6), and in equations (1.1) and (1.2) Γ must be substituted for $-\Gamma$. In order for the

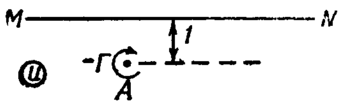

Fig. 4.

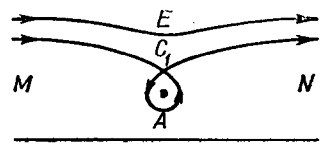

Fig. 5.

flow to exist it is necessary, as noted above, that $\kappa^2 > 1$. Replacing the circulation $-\Gamma$ by Γ in the expression for κ and transforming it, we find

$$\varkappa = \frac{(\gamma + \gamma^{-1})^2}{2 + \Gamma q^{-1}(\gamma^{-1} - \gamma)} - 1 \tag{5.1}$$

Since $0 < \gamma < 1$, it is evident that κ may not be smaller than -1, and therefore, a solution of the problem is possible only for $\kappa > 1$. Substituting κ from (5.1) into this inequality, we obtain the inequality

$$\gamma^{-1} - \gamma > 2\Gamma / q \tag{5.2}$$

In the limiting case the inequality (5.2) transforms into an equality, and we obtain from it

$$\gamma = \sqrt{\Gamma^2 / q^2 + 1} - \Gamma / q, \qquad \delta = i$$

This limiting case is not possible because the critical point cannot come up to the free surface.

Using (2.3) we may obtain the smallest possible value of vortex immersion. In the case under consideration the force Y (see (3.2)), acting on the vortex, will be not a lift force but a downward force. When the vortex approaches the bottom Y approaches $-\infty$. The value of Y, when point A approaches its highest location, approaches the value

$$-\rho v_0 \Gamma (2 + \Gamma^2 / q^2)$$

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